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The solution of contact problems using boundary element method☆

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Abstract

The formulation of contact problems is extended to the case of moving punches and to the case when the state of the systems being investigated depends on the history of the change in the external actions. The quasi-static contact problem for a moving rigid rough punch and a single linearly deformable body is considered. A new iterational process is proposed for solving contact problems, taking friction in the contact area into account, and its convergence is proved. An algorithm of the solution, based on the boundary element method, is developed. Solutions of specific problems are given and analysed. Estimates of the difference of the solutions due to the difference in the impenetrability conditions and the difference in the steps of the loading parameter are obtained.

The variational approach used to solve contact problems was considered for the first time by Signorini in Refs. 1,2. The results of the development of this stage in solving contact problems has been presented by Lions and Duvaut,[3] who also considered contact problems with friction on the assumption that the Amonton-Coulomb friction law is related to the displacement rather than to the relative slip velocity. This limitation was later removed.[4]

The advantage of the variational approach, in addition to the possibility of proving existence and uniqueness theorems, is the possibility of constructing numerical methods of solution that are effective in practice and justified theoretically. The first fundamental generalizing monograph in this area was Ref. 5. Numerical solutions of contact problems using relative velocities and the Amonton-Coulomb friction law were given for the first time in Ref. 6.

An extension of the results to the case of contact problems, taking adhesive stick into account was developed in Refs. 7–9, and numerical and analytical solutions of problems taking adhesion into account, having important applications in nanomechanics, were given in Ref. 10 (see also the monograph Ref. 11). Analytic solutions of these problems were obtained in Ref. 12.

Analytic solutions are important for a qualitative analysis of the effects related to contact friction forces. Naturally, the construction of accurate analytic solutions requires the introduction of additional hypotheses both as regards the geometry of the region (canonical regions – a half-space, a half-plane, a circle, etc. are usually considered), and as regards the friction laws; the relation between the contact pressure and the shear force in the contact region is most often specified,[13] or the parameters of the interface of the stick and slip regions are introduced as additional unknowns (see Refs. 14–16, etc.).

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1. Local formulation of the problem: fundamental notation and relations

We will consider the problem of the contact between a deformable body, occupying the region Ω with boundary $\Sigma = \partial\Omega$, and a rigid rough punch. The main hypotheses and relations are the same as those used previously in Ref. 11, with the exception of the hypotheses that the punch is fixed and regarding the possibility of linearizing the impenetrability condition.

We will use the following notation: $\hat{\sigma}$ is the stress tensor, where the "hat" indicates a second-rank tensor, $\hat{\varepsilon} = (\nabla\mathbf{u} + \nabla\mathbf{u}^T)/2$ is the tensor of small Cauchy strains, in a Cartesian system of coordinates: $\varepsilon_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2$, $\mathbf{u}$ is the displacement vector, $\boldsymbol{\nu}$ is the vector of the unit outward normal to the boundary Σ, $\sigma = \hat{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}$ is the traction vector on the boundary Σ, $\boldsymbol{\sigma} = \sigma_N\boldsymbol{\nu} + \boldsymbol{\sigma}_T$ is the decomposition of the surface traction vector into normal and tangential (i.e. tangential to the boundary) components, f is the friction coefficient, and x are the coordinates of points in space.

We will consider the quasi-static problem, neglecting forces of inertia; the extension to dynamic problems is constructed using the results obtained previously.[17] The parameter which determines the change in the state of the "body-punch" system will be denoted by t; in dynamics, t is the time, but in the problems considered here we choose as the parameter t the depth of penetration of the punch.

The local formulation of the contact problem contains the equilibrium equations (for simplicity, mass forces are assumed to be equal to zero) and Hooke's law

$$\nabla \cdot \hat{\boldsymbol{\sigma}} = 0; \quad \hat{\boldsymbol{\sigma}} = {}^4\hat{a} \cdot\cdot\, \hat{\boldsymbol{\varepsilon}} \tag{1.1}$$

where ${}^4\hat{a}$ is the tensor of the moduli of elasticity, and the superscript 4 indicates the fourth rank. Note that, since the boundary element method is being used below, to construct the resolvent it is necessary to know the fundamental solution corresponding to this tensor ${}^4\hat{a}$. For simplicity, we will assume below that the body is isotropic and homogeneous and we will use Kelvin's solution (although solutions for certain types of anisotropy and some forms of inhomogeneity are known at the present time).

We will add the following boundary conditions – in terms of forces on the part $\Sigma_\sigma \subset \Sigma$, to Eq. (1.1)

$$\hat{\boldsymbol{\sigma}}\mathbf{v}|_{\Sigma_\sigma} = \mathbf{P}(\mathbf{x}, t), \quad \mathbf{x} \in \Sigma_\sigma \tag{1.2}$$

and in the displacement on the part $\Sigma_u \subset \Sigma$

$$\mathbf{u}|_{\Sigma_u} = \mathbf{U}(\mathbf{x}, t), \quad \mathbf{x} \in \Sigma_u \tag{1.3}$$

where $\mathbf{P}(\mathbf{x}, t)$ and $\mathbf{U}(\mathbf{x}, t)$ are specified functions.

By assumption, $\Sigma = \Sigma_\sigma \cup \Sigma_u \cup \Sigma_C$, where Σ_C is the part of the boundary of the deformable body, points of which may be in contact with the punch.

To formulate the boundary conditions on the part of the surface Σ_C, we will introduce a fixed (laboratory) system of coordinates $Oxyz \equiv Ox_1x_2x_3$ and a system of coordinates $O_1\xi_1\xi_2\xi_3$, rigidly connected with the moving punch; both systems, by assumption, are Cartesian and, at the initial instant of time, coincide (although this is not necessary). The equation of the boundary (surface) of the punch at the initial instant of time will be taken in the form

$$\Psi(\boldsymbol{\xi}) = 0 \tag{1.4}$$

By assumption, $\Psi(\xi) > 0$ for points with coordinates $\boldsymbol{\xi}$ outside the region occupied by the punch, and $\Psi(\xi) < 0$ for points inside the punch.

We will move the punch translationally along the vector $\mathbf{U}_p$, and then we will rotate it in space, specifying this rotation by the Euler angles between the axes of the moving and fixed systems of coordinates. Defining the rotation matrix $\hat{A}$ in the usual way, we will write the relation between the coordinates of the same point of the punch surface before and after deformation in the following form

$$\mathbf{x} = \mathbf{U}_p + \hat{A} \cdot \boldsymbol{\xi} \tag{1.5}$$

Obviously, in the case when the state of the system being investigated depends on the history of the change in the external actions, the quantities $\mathbf{U}_p$ and $\hat{A}$ must be specified as functions of the time t (which is also done in the present paper). Note that, for continuously varying values of $\mathbf{U}_p$ and $\hat{A}$, we must use formula (1.5) for infinitesimal changes. Considering Eq. (1.5) as an equation in $\boldsymbol{\xi}$, we obtain

$$\boldsymbol{\xi} = \hat{A}^{-1} \cdot (\mathbf{x} - \mathbf{U}_p) \tag{1.6}$$

The position of an arbitrary point $\mathbf{x} \in \Sigma_C$ after deformation will be defined by the radius vector $\mathbf{x} + \mathbf{u}(\mathbf{x}, t)$; from formula (1.6) and the hypothesis regarding the function ξ we obtain the first of the relations on the part Σ_C of the boundary

$$\Psi[\hat{A}^{-1} \cdot (\mathbf{x} + \mathbf{u}(\mathbf{x}, t) - \mathbf{U}_p)] \geq 0, \quad \forall \mathbf{x} \in \Sigma_C \tag{1.7}$$

which reflects the requirement that no points of the deformable body penetrate into the punch.

By assumption, there are no stretching normal forces, and hence,

$$\sigma_N(\mathbf{x}, t) \leq 0, \quad \forall \mathbf{x} \in \Sigma_C \tag{1.8}$$

From physical considerations we have the equation

$$\Psi[\hat{A}^{-1} \cdot (\mathbf{x} + \mathbf{u}(\mathbf{x}, t) - \mathbf{U}_p)]\sigma_N(\mathbf{x}, t) = 0, \quad \forall \mathbf{x} \in \Sigma_C \tag{1.9}$$

sometimes called the complementarity condition.

The complete set of relations on the part Σ_C of the boundary contains the laws which model the behaviour of the shear components of the forces and displacements. Thus, when there is no friction, we will have

$$\boldsymbol{\sigma}_T(\mathbf{x}, t) = 0, \quad \forall \mathbf{x} \in \Sigma_C \tag{1.10}$$

We will also use Coulomb's friction law, according to which we will have the relations

$$\begin{aligned} &|\boldsymbol{\sigma}_T| \leq f|\sigma_N| \Rightarrow \dot{\mathbf{u}}_T = 0 \\ &|\boldsymbol{\sigma}_T| = f|\sigma_N| \Rightarrow \exists \kappa \geq 0 : \dot{\mathbf{u}}_T = -\kappa \boldsymbol{\sigma}_T \end{aligned} \tag{1.11}$$

where f is the friction coefficient. If slip occurs, then $|\dot{u}_T| > 0$, and

$$\boldsymbol{\sigma}_T / |\boldsymbol{\sigma}_T| = -\dot{\mathbf{u}}_T / |\dot{\mathbf{u}}_T|$$

The quantity $\dot{\mathbf{u}}_T$ is equal to the relative rate of slip of points of the body along the punch, i.e. the derivative of the tangential component of the relative displacement vector with respect to the parameter t.

2. Variational formulation and method of solution

The change to a variational formulation was first made in Ref. 4, subsequently reproduced in Ref. 11 and extended to dynamic problems in Ref. 17. The main result for the quasi-static problem considered here can be formulated in the form of the following theorem.

Theorem 1. *The solution of problem (1.1)–(1.11) is equivalent to the solution of the variational inequality*

$$a(\mathbf{u}, \delta\dot{\mathbf{u}}) + \int_{\Sigma_c} f|\sigma_N(\mathbf{u})|(|\dot{\mathbf{v}}_T| - |\dot{\mathbf{u}}_T|)d\Sigma \geq L(\delta\dot{\mathbf{u}}), \quad \forall \delta\dot{\mathbf{u}} = \dot{\mathbf{v}} - \dot{\mathbf{u}}, \quad \dot{\mathbf{v}} \in \dot{K}_u \tag{2.1}$$

where

$$a(\mathbf{u}, \delta\mathbf{u}) = \int_{\Omega} \hat{\sigma}(\mathbf{u}) \cdot\cdot\, \hat{\varepsilon}(\delta\mathbf{u}) d\Omega, \quad L(\delta\mathbf{u}) = \int_{\Omega} \rho\mathbf{F} \cdot \delta\mathbf{u} d\Omega + \int_{\Sigma_\sigma} \mathbf{P} \cdot \delta\mathbf{u} d\Sigma$$

$$\dot{K}_u = \{\dot{\mathbf{v}} | \dot{\mathbf{v}} = \dot{\mathbf{u}} + \delta\dot{\mathbf{u}};\ \Psi(\alpha)'_\alpha \cdot (\hat{A}^{-1} \cdot \delta\dot{\mathbf{u}}) \geq 0,\ \forall \mathbf{x} \in \Sigma_C^t\}$$

$$\Sigma_C^t = \left\{\mathbf{x} | \mathbf{x} \in \Sigma_C;\ \Psi(\alpha(\mathbf{x})) = 0;\ \hat{A}' \cdot (\dot{\hat{A}}^{-1} \cdot (\mathbf{x} - \mathbf{U}_p + \mathbf{u}) + \hat{A} \cdot (-\hat{\mathbf{U}}_p + \dot{\mathbf{u}})) = 0\right\} \tag{2.2}$$

$$\hat{A} = \hat{A}(\alpha), \quad \alpha = \hat{A}^{-1} \cdot (\mathbf{x} - \mathbf{U}_p + \mathbf{u})$$

$\dot{\mathbf{u}}$ *is the derivative of the exact solution* $\mathbf{u}$ *with respect to the parameter* t, *and a prime denotes a derivative with respect to the variable* α.

The proof of the theorem repeats, with some changes, the proof of the corresponding theorems given previously in Refs. 4,11,17. The main new problem consists of constructing kinematically admissible velocity fields for the case of a moving punch; it is solved by Ostrogradskii's method.[18] According to this method, as it applies to the quasi-static problem considered here, at a given instant of time t one must bear in mind the connections solely at those points of the surface Σ_C at which, first, intersection with the punch occurs – the impenetrability condition is satisfied with a strict equality sign, and second, the velocity of motion of a particle of the body at a given point is identical with the velocity of motion of a point on the punch surface. It is precisely these two limitations that also occur in the definition of the set Σ_C^t.

As regards the determination of the admissible velocities, according to Ostrogradskii, these velocities are quite arbitrary (naturally, they differ by an infinitesimal amount from the true values) at those points at which there is no contact, i.e. the impenetrability condition is satisfied with a strict inequality sign; at contact points the admissible velocities can be directed only outside the punch. This principle is the basis for determining the set $\dot{K}_u$. A discussion of the problems that arise on changing to dynamic problems can be found in the commentaries to Ostrogradskii's paper (Ref. 18, pp. 346–352).

Note that, in the first paper,[6] devoted to the contact problem with friction, instead of possible velocities, the true and possible (virtual) increments of the displacements were used when changing from the instant of time t to the instant $t + dt$. This approach enabled a step-by-step algorithm to be constructed for taking into account the loading history of the system, to formulate an iterational algorithm for taking the mutual effect of the friction forces and the contact pressure into account (see below) and to prove its convergence. At the same time, the introduction of admissible velocities enables the relation between the approach developed and the classical Jordan variational principle to be revealed, and also enables the transfer to a dynamic problem to be made.

Note that inequality (2.1) cannot be reduced to the problem of finding the stationary point of a certain functional, and the set of velocities that it is admissible to compare depends on the solution. Inequalities of this kind are equivalent (of course, in a generalized sense) to problems with unilateral constraints, and have been called, in Lions' papers, quasi-variational problems.

The method of solution is constructed as follows. There is a theorem based on the necessary and sufficient conditions for a convex functional to have a minimum on a closed convex set K.[11] The theorem is as follows.

Theorem 2. *If the contact pressure* $\sigma_N(\mathbf{u}) \equiv \mathcal{F}$ *in quasi-variational inequality* (2.1) *is known, while the relative velocities in the Amonton-Coulomb friction law is replaced by relative displacements, this inequality is equivalent to the problem of minimizing the functional*

$$J(\mathbf{v}) = \frac{1}{2}a(\mathbf{v}, \mathbf{v}) - L(\mathbf{v}) + \int_{\Sigma_C} f|\mathcal{F}||\mathbf{v}_T| d\Sigma \to \min_{\mathbf{v} \in K_u} \tag{2.3}$$

where K_u *is the set of kinematically admissible displacement fields which satisfy the impenetrability condition* (1.7), *in which the quantity* $\mathbf{u}(\mathbf{x}, t) - \mathbf{U}_p$ *is replaced by the quantity* $\mathbf{v}(\mathbf{x}) - \mathbf{U}_p$.

We will now return to the initial formulation of the Amonton-Coulomb law (in velocities). To take into account the loading history, we will consider the problem of finding the increments of the displacements $d\mathbf{u} = \mathbf{u}^{t+dt} - \mathbf{u}^t$, $\mathbf{u}^t = \mathbf{u}(\mathbf{x}, t)$ – the displacement field for the instant of time t, where $\mathbf{u}^{t+dt} = \mathbf{u}(\mathbf{x}, t + dt)$ is the displacement field for an infinitely close instant of time $t + dt$. We will denote the total increment of the displacements after a time dt by $\delta * \mathbf{u}^t$, and we will denote the linear part of this displacement by $d\mathbf{u}^t$, but we will keep the notation δ for the variation. By definition,

$$\begin{aligned}&\delta^*\mathbf{u}^t = \mathbf{u}^{t+dt} - \mathbf{u}^t, \quad \delta\mathbf{u}^t = \mathbf{w} - \mathbf{u}^t\\&d\mathbf{u}^t = \dot{\mathbf{u}}(\mathbf{x}, t)dt, \quad d\mathbf{v}^t = \dot{\mathbf{v}}(\mathbf{x}, t)dt\end{aligned} \tag{2.4}$$

where $\mathbf{w}$ is the admissible displacement field for the instant of time $t + dt$, $\dot{\mathbf{u}}(t)$ is the field of true velocities at the instant t, and $\dot{\mathbf{v}}(t)$ is the field of admissible velocities. Replacing the velocities in inequality (2.1) using formulae (2.4), using the transformation $\mathbf{w} - \mathbf{u}^{t+dt} = (\mathbf{w} - \mathbf{u}^t) - (\mathbf{u}^{t+dt} - \mathbf{u}^t)$ and the hypotheses: a) the contact pressure is known, and b) $\mathbf{w} = \dot{\mathbf{v}}dt$, we arrive at the problem (see Ref. 5):

$$J(\mathbf{v}) = \frac{1}{2}a(\mathbf{v}, \mathbf{v}) - L(\mathbf{v}) + \int_{\Sigma_C} \mathcal{F}\left|\mathbf{v}_T - \mathbf{u}_T^t\right| d\Sigma \to \min_{\mathbf{v} \in K_u} \tag{2.5}$$

We have the following limit equality

$$f\left|\sigma_N^{(r)}\right|\left|\mathbf{v}_T - \mathbf{u}_T^t\right| = \max_{\boldsymbol{\mu}_T, |\boldsymbol{\mu}_T| \le f\left|\sigma_N^{(r)}\right|} [\boldsymbol{\mu}_T \cdot (\mathbf{v}_T - \mathbf{u}_T^t)] \tag{2.6}$$

An elementary (non-rigorous) proof of this inequality of the theorem is obvious: the scalar product of two vectors, one of which is fixed, while the second does not go beyond the limits of a circle of constant radius, is a maximum when these vectors are parallel, and the end of the variable vector is on the boundary of the circle. A rigorous proof, in which it is taken into account that generalized solutions occur in variational problems, can be found in many publications (see, for example, Ref. 5).

Suppose now the contact pressure $\sigma_N \equiv \mathcal{F}$ is the required independent function as well as the friction forces $\boldsymbol{\sigma}_T$. By using the assertions and results discussed in Ref. 11, we can establish that the problem in question is equivalent to the problem of finding the saddle point

$$\frac{1}{2}a(\mathbf{v}, \mathbf{v}) - L^{(t+dt)}(\mathbf{v}) + \int_{\Sigma_C} [\boldsymbol{\sigma}_T \cdot (\mathbf{v}_T - \mathbf{u}_T^t) + \sigma_N(\delta_N - \upsilon_N)] d\Sigma \to \min_{\mathbf{v} \in V} \max_{\sigma_N \le 0} \max_{|\boldsymbol{\sigma}_T| \le f|\sigma_N|} \tag{2.7}$$

in which the quantity $\mathbf{u}_T^t$ is known and corresponds to the state of the external action for the instant of time t (the superscript $t + dt$ is omitted).

A rigorous justification of this procedure requires proof of the existence theorem for the whole loading process.

We used Uzawa's method: the motion to the critical point in the direction of steepest descent of a functional with respect to the variable $\mathbf{v}$ and the steepest ascent with respect to the variables σ_N, $\boldsymbol{\mu}_T$, with return, if necessary, to the set of admissible values of σ_N, $\boldsymbol{\mu}_T$ along the shortest path. The convergence of this iteration process is ensured by the strict convexity of functional (2.7) with respect to the variable $\mathbf{v}$ and concavity (non-rigorous) with respect to the variables $(\boldsymbol{\sigma}_N, \boldsymbol{\sigma}_T)$.

Uzawa's method contains the following steps:

1) a certain zero approximation $\sigma_N^{(r+1)(0)}$, $\boldsymbol{\sigma}_T^{(r+1)(0)}$ is chosen for the contact interaction forces;
2) a minimum of the functional (2.7) with respect to the displacements $\mathbf{v}$ is found, which is equivalent to the solution of the standard problem of the theory of elasticity (without constraints on the form of the inequalities) with the following boundary conditions on Σ_C

$$\sigma_{ij}\nu_j\big|_{\Sigma_C} = \sigma_N^{(r+1)(0)}\nu_i + (\sigma_T^{(r+1)(0)})_i \tag{2.8}$$

(the result is the displacement field $\mathbf{u}^{(r+1)(0)}$);
3) using this field we find the corresponding distribution of the contact interaction forces

$$\sigma_N^{(r+1)(1)} = P_N(\sigma_N^{(r+1)(0)} + \rho_{0N}\Delta^{(r+1)})$$

$$\boldsymbol{\sigma}_T^{(r+1)(1)} = P_T(\boldsymbol{\sigma}_T^{(r+1)(0)} + \rho_{0T}(\mathbf{u}_T^{(r+1)(0)} - \mathbf{u}_T^t))$$

where we have introduced orthogonal projections of the corrected contact interaction forces on the admissible set

$$P_N(\sigma_N) = \begin{cases} \sigma_N, & \sigma_N \le 0 \\ 0, & \sigma_N > 0 \end{cases}, \quad P_T(\boldsymbol{\sigma}_T) = \begin{cases} \boldsymbol{\sigma}_T, & |\boldsymbol{\sigma}_T| \le f|\sigma_N^{(0)}| \\ \dfrac{\boldsymbol{\sigma}_T}{f|\boldsymbol{\sigma}_T|}|f||\sigma_N^{(0)}|, & |\boldsymbol{\sigma}_T| > f|\sigma_N^{(0)}| \end{cases}$$

determined, as already mentioned, by the condition for there to be no tensile contact stresses and Coulomb's friction law; ρ_{0N} and ρ_{0T} are numerical parameters, which control the rate of convergence. The quantity $\Delta^{(r+1)}$ is a measure of the gap between the contacting bodies in the current iteration, for example, $\Delta^{(r+1)} = (\delta_N - u_N^{(r+1)(0)})$, or, in accordance with relation (1.7),

$$\Delta^{(r+1)} = \Psi[\hat{A}^{-1} \cdot (\mathbf{x} + \mathbf{u}^{(r+1)}(\mathbf{x}, t) - \mathbf{U}_p)]$$

In this research we also used a measure of the gap introduced in the paper by Hertz (see below). Note that in the formula for the correction of the friction forces P_T we can use the value of the normal pressure already corrected in the current iteration.

3. Discretization

We will begin with discretization with respect to the parameter t, which defines the manner in which the external actions change. We will construct a solution $\mathbf{u}(\mathbf{x}, t)$ for values of $t \in [0, T]$. We will divide the interval $[0, T]$ by points (nodes) t_k, assuming $t_0 = 0$, $t_N = T$, $\Delta t_k = t_{k+1} - t_k$. We will denote the solution at the point $t = t_k$ by $\mathbf{u}^k$, and at the point $t = t^{k+1}$ by $\mathbf{u}^{k+1}$. We will assume that the initial value $\mathbf{u}^0$ is specified (usually the natural stress- and strain-free state is chosen as $\mathbf{u}^0$). We then obtain the following quasi-variational inequality for finding the solution for the instant of time $t = t_{k+1}$

$$a(\mathbf{u}^{k+1}, \mathbf{v} - \mathbf{u}^{k+1}) - L(\mathbf{v} - \mathbf{u}^{k+1}) + \int_{\Sigma_C} f|\sigma_N(\mathbf{u}^{k+1})|(|\mathbf{v}_T - \mathbf{u}_T^k| - \\ - |\mathbf{u}_T^{k+1} - \mathbf{u}_T^k|)d\Sigma \ge 0 \;\; \forall \mathbf{v} \in K, \; \mathbf{u}^{k+1} \in K \tag{3.1}$$

the solution of which is constructed using the iteration process described above.

We will use the boundary-elements method for discretization with respect to the spatial variable.

Its advantage compared with the finite element method is the fact that here the condition of equilibrium inside the region Ω and the boundary conditions on the boundary Σ are satisfied exactly, which enables us to construct approximate solutions with high accuracy with comparatively low requirements on the memory and speed of response of the computer; the use of the boundary element method also leads to a reduction in the number of iterations (compared with the finite element method) in the successive-approximation processes considered, particularly when solving contact problems in which friction is taken into account. Moreover, the singularities of the solution, which arise due to the presence of corner points and points of change of the boundary conditions type, can be comparatively easily recognised.

The discretization algorithm based on the boundary element method has been described in many publications. Here we will use the version based on Somigliana's formula[19]

$$\mathbf{u}(\xi) = \int_\Sigma \hat{u}^*(\boldsymbol{\xi}, \mathbf{x}) \cdot \mathbf{p}(\mathbf{x}) d\Sigma - \int_\Sigma \hat{p}^*(\boldsymbol{\xi}, \mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) d\Sigma \tag{3.2}$$

where $\hat{u}^*(\boldsymbol{\xi}, \mathbf{x})$ and $\hat{p}^*(\boldsymbol{\xi}, \mathbf{x})$ are the matrices of the fundamental solutions, and $\mathbf{p}(\mathbf{x})$ and $\mathbf{u}(\mathbf{x})$ are the boundary values of the forces and displacements respectively. In the case of an isotropic body with a shear modulus G and a Poisson's ratio ν, we have for the plane-deformation problems considered in this paper

$$\hat{u}^*(\boldsymbol{\xi}, \mathbf{x}) = -\frac{1}{8\pi(1-\nu)G}[(3-4\nu)\ln r\hat{\delta} - \nabla r \otimes \nabla r] \tag{3.3}$$

$$\hat{p}^*(\boldsymbol{\xi}, \mathbf{x}) = -\frac{1}{4\pi(1-\nu)r}\{[(1-2\nu)\hat{\delta} + 2\nabla r \otimes \nabla r]\nabla r \cdot \mathbf{n} - (1-2\nu)(\nabla r \otimes \mathbf{n} - \mathbf{n} \otimes \nabla r)\} \tag{3.4}$$

where $\hat{\delta}$ is the unit tensor, and $\otimes$ is the operation of dyadic multiplication (the vector of the unit outward normal is denoted here by $\mathbf{n}$, in order not to confuse it with Poisson's ratio ν).

Taking the limit $\xi \to \mathbf{x} \in \Sigma$, we arrive at a boundary integral equation of the theory of elasticity, which can be found, for example, in Ref. 19. The boundary element method equations are obtained from this integral equation by using a piecewise-linear approximation of the functions $\mathbf{p}(\mathbf{x})$, $\mathbf{u}(\mathbf{x})$, $\mathbf{x} \in \Sigma$ and employing double nodes on the boundary, which enables the presence of corner points and points of change of the boundary conditions type to be taken into account.

4. Examples of numerical solutions and their analysis. Description of the problem

We will consider the problem of the indentation of a rigid circular punch into a rectangular region (see Fig. 1).

The side DA of the rectangle $ABCDA$ is rigidly clamped while the sides AB and CD are force-free. The surface Σ_C in the problem considered is the section BC. The punch moves vertically downwards along the Oy axis, situated symmetrically with respect to the rectangle, and since the centre of the circle lies on the Oy axis, the problem is symmetrical about this axis; we will use the quantity U_y, shown in Fig. 1, as the parameter t. Hence, in condition (1.7)

$$\hat{A} = \hat{A}^{-1} = \hat{\delta}, \quad \mathbf{U}_p = (0, U_y), \quad y = AB \tag{4.1}$$

Consequently,

$$\Psi(\mathbf{x}, \mathbf{u}, \mathbf{U}_p) = \Psi(((x, y), (u_x, u_y), (U_x, U_y))_{\Sigma_C} = (x + u_x)^2 + (U_y + u_y - R)^2 - R^2 \geq 0 \tag{4.2}$$

In the classical formulation of problems of the contact of deformable bodies, attributed to Hertz, the linearized impenetrability condition is used, where it is assumed that points of the boundaries of the contacting bodies move parallel to the normal to the common tangential plane at the initial point where they touch (we recall that in Hertz paper, devoted to an analysis of the contact of glass lenses, the bodies were assumed to be convex, and there was only one initial point where they touched). The analytical form of the Hertz condition is

$$U_y + u_y - R + \sqrt{R^2 - x^2} \geq 0 \tag{4.3}$$

Note that this hypothesis (or one close to it) is used in practically all the contact problems investigated; note also that a theoretical analysis of the problem of the relation between different forms of the impenetrability conditions was presented previously in Ref. 11. In the case when one of the bodies (the circular punch in the problem considered) is rigid, a more accurate formulation will be one in which projections of the displacements and contact forces onto the normal and tangent to the punch will be more accurate. Below we will give the results of a numerical investigation.

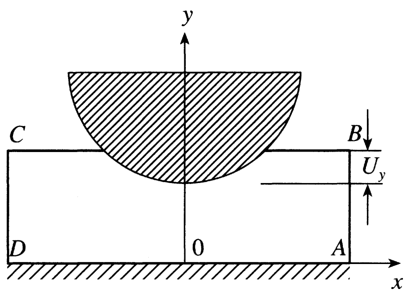

Fig. 1.

It is important to note that the use of the exact impenetrability condition (without linearization) enables one to investigate the features of the solution at corner points of the boundary and problems with non-smooth punches. However, the exact impenetrability condition may lead to certain problems when investigating the uniqueness of the solution; this fact, first pointed out by Vorovich, is related to the possible dependence of the solution on the loading history, even in problems in which friction is ignored. The linearized impenetrability conditions are used when proving certain theorems on the equivalence of the local and variational formulations of contact problems, and also in theorems on the uniqueness of the solution for non-convex punches.

In many papers on contact problems in which friction is taken into account, the relative velocities in Coulomb's friction law are replaced by displacements. Numerical solutions of problems of the contact of deformable bodies with friction using velocities, including more complex loading processes, were given for the first time using the example of the axisymetric problem of the indentation of a sphere into a half-space,[4] in which the linearized impenetrability condition was used. Below we will present a comparison of the solutions of the problems for one and several steps of the parameter t (one step corresponds to the use of the displacements in the friction law).

The results obtained, illustrated by the curves in the figures presented below, correspond to the following initial data: the number of boundary elements on the sides AB and CD is equal to 50, on the sides BC and DA it is equal to 200, Young's modulus is equal to 5 (to increase the orders of the deformations and displacements, although similar results were also obtained for practical materials like tempered steel when investigating the contact interaction of a railroad wheel with the rail), Poisson's ratio is equal to 0.3, the maximum displacement of the punch along the vertical is equal to $0.1AB = 0.5BC$, $AB = 1$ and $R = 4$, the steps in t were chosen to be same, the friction coefficient $f = 0.2$, and the criterion for the completion is that the difference in the Euclidean norm of two iterations with respect to the displacements (in the contact region) does not exceed 10^{-7}.

Analysis of the numerical results. The most interesting and important new results are related to the choice of the directions of the projection of the contact interaction forces. As already noted, in the case of a rigid punch being considered here, the calculation of the contact pressure by projection onto the normal to the punch is physically justified (in the calculations we used the inward normal), as well as the calculation of the friction force by projection onto the tangent to the punch (we will call this method Version a). At the same time, in the absolute majority of papers, a formulation of the problem is used when, by definition, the contact pressure is directed along the normal to the common tangent plane (in the problem considered, this means along the normal to the section BC), while the friction forces are directed along the tangent (along BC, Version b), while the effect of the friction forces on the normal pressure is neglected. As it turned out, in all cases the solution depends on the number of steps with respect to the loading parameter.

The results obtained are shown in Fig. 2 in the form of graphs of the friction forces P_T as a function of the variable s, equal to the distance of the current point (a node of the grid of boundary elements) on the side BC from the point B, and along the horizontal axis we have plotted the number of the nodes I. In all there are 200 points, and all the curves are shown for 100 points, since the pattern is antisymmetric about the point $I = 100$. The continuous curves correspond to the formulation of the problem in Version a for different values of the depth of penetration of the punch U_y. The dashed and dash-dot curves were obtained for Version b with $U_y = 0.1$, in which case the effect of the friction forces on the normal pressure was ignored. The dash-dot curve is obtained for the formulation of the problem when, first, the

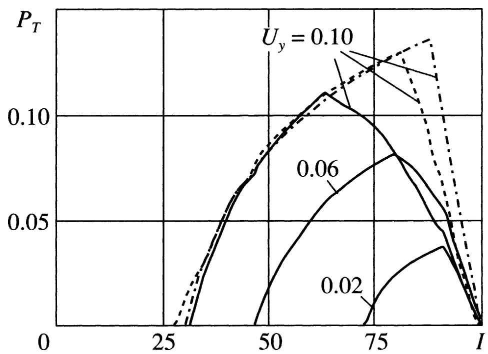

Fig. 2.

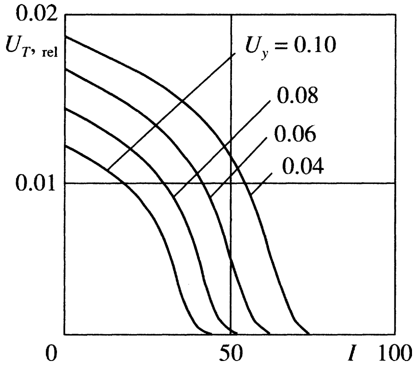

Fig. 3.

contact stresses are projected onto the normal to the section BC, and, second, when there is only one step along the imbedding; the dashed curve is obtained for 5 steps. It can be seen that the difference in the solutions in Versions a and b may reach 25%.

Using the example of the problem in Version a we investigated the problem of the effect of the choice of the impenetrability conditions on the solution – we used the exact non-linear impenetrability condition (4.2) and the Hertz condition (4.3). As might have been expected, the limit of the sequence of approximate solutions turned out to be one and the same, but the number of iterations and, consequently, the operating time of the processor when choosing the Hertz condition (4.3) was approximately four times greater than for condition (4.2).

We also point out that the distribution of the contact pressure and of the vertical displacements in the contact area when there is no friction are practically identical for the exact and linearized impenetrability conditions: the difference in the displacements does not exceed 0.1%, and the difference in the contact pressure does not exceed 2%. However, the difference in the horizontal displacements can be considerable – in the problem considered the maxima of the horizontal displacements differed by 30%.

In order to verify that a maximum of the friction forces is reached at the interface of the stick and slip regions, and also to investigate the evolution of the stick region as the depth of indentation of the punch increases, we obtained the differences $U_{T,\,\mathrm{rel}}$ of the vertical displacements of the boundary for the current and previous steps. The curves obtained are shown in Fig. 3 (the curves are symmetrical about the middle of the contact area, and hence we only show a half of each of them). As in Fig. 2, along the horizontal axis we have plotted the number of the node on the boundary BC, measured from the point B. The horizontal parts of the curves, which extend monotonically as U_y increases, represent the stick regions; the end points of these regions correspond to the maximum values of the friction forces.

The results obtained confirm that, when solving problems taking Amonton-Coulomb friction into account, one must use the exact impenetrability condition, and project the forces and displacements of the boundary onto the normal and tangent to the rigid punch. Moreover, one must use multistep procedures with respect to the loading parameter, in which case the effect of friction forces on the contact pressure may be ignored. The above factors, when taken into account, may lead to a serious correction when estimating the value of the wear as a result of friction and, correspondingly, of the working life of contacting parts of articles.

Note that we have chosen as the zero approximations in problems without friction Sadovskii's solution on the interaction of a rigid punch with an elastic half-plane.[13] The zero approximations for the normal pressure in problems with friction at all stages were constructed in the same way as in problems without friction. The zero approximations for the friction forces were assumed to be either equal to zero or proportional to the normal pressure; some other versions were also tested. It turned out that, in all cases, the successive approximations converged to the same limit, but the number of iterations depends on the choice of the zero approximation: the closer the zero approximation to the exact solution, the less the number of iterations required to achieve a specified accuracy. Some test calculations showed that the number of iterations can be reduced by the optimum choice of the parameters in the formulae for correcting the shear and normal forces; the results presented were obtained for $\rho_{0N} \equiv \rho_0 = \mathrm{const} = 1$, $\rho_{0\mathrm{T}} \equiv \rho_0 = \mathrm{const} = 1$ (which are not optimum).

Using Il'yushin's method of elastic solutions, the above approach can be extended to problems of the deformation theory of plasticity.[20] In this case, fictitious mass forces arise, the effect of which for the numerical realization is taken into account by splitting the calculation region into finite elements and subsequently evaluating integrals, the integrands of the expressions in which are the products of the finite-element approximations of the fictitious mass forces and the elements of the matrix of Kelvin's fundamental solution.

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